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IMPLEMENTATION OF SERVO SYSTEMS FOR CONTROLLING A DOUBLE PURPOSE NONLINEAR PLANT: INVERTED PENDULUM AND CRANE

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RESUMEN

Este artículo desarrolla procedimientos de diseño de Servosistemas tipo Proporcional Integral (SS abreviado) para controlar una planta no lineal de doble propósito: el Péndulo Invertido y Grúa (PI&G). Dicha planta se puede describir mediante ecuaciones diferenciales no lineales, donde los términos no lineales complican los aspectos de modelado y de diseño del controlador. Sin embargo, basado en el modelo lineal de la planta, nosotros podemos configurar SSs discretos combinando controladores con observadores no lineales. Resultados experimentales demuestran que cada SS es capaz de estabilizar el PI (o Grúa) montado sobre un carro movido por un servomotor. Tres configuraciones de SSs se presentan y discuten: un Controlador Óptimo Cuadrático de Estado Estable (COCEE) con un Observador de estado Óptimo Cuadrático (OEOC), un Controlador por Ubicación de Polos (CUP) con un Observador de Estados de Orden Completo (OEOC), y un CUP con un Observador de Estados de Orden Mínimo (OEOM).

ABSTRACT

This paper develops design procedures of Proportional-Integral Servo Systems (SSs for short) for controlling a double purpose nonlinear plant: Inverted Pendulum and Crane (IP&C). Such a plant can be described by nonlinear differential equations, where the nonlinear terms complicate the analytical aspects of modeling and controller design. However, based on the linearized plant model, we can configurate discrete-time SSs by combining linear controllers with linear observers. Experimental results demonstrate that each SS is able to stabilize an IP (or a Crane) mounted on a servomotor-driven cart. Three SS configurations are developed: A Steady-State Quadratic Optimal Controller (SSQOC) with a Quadratic Optimal State Observer (QOSO), a Pole Placement Controller (PPC) with a Full-Order State Observer (FOSO), and a PPC with a Minimum-Order State Observer (MOSO).

MODELING THE IP&C PLANT

Fig. 1 depicts the nonlinear IP plant, which can be described by the following nonlinear model [1]

$$M_1\ddot{z} - M_2(\operatorname{sen}\theta)\dot{\theta}^2 + M_2(\cos\theta)\ddot{\theta} - F = 0 \quad (1)$$

$$M_2 g(\operatorname{sen}\theta) - M_2\ddot{z}(\cos\theta) - J_1\ddot{\theta} = 0 \quad (2)$$

$$F = K_x K_A u - J_2\ddot{z} - B_x\dot{z} \quad (3)$$

where:

$$M_1 = m_c + M_e + m_v \quad ; \quad M_2 = m_e l_e + m_v \frac{l_v}{2}$$

$$J_1 = J_e + J_v \quad ; \quad J_2 = \frac{J_{eq}}{n^2 r_p^2}$$

$$K_x = \frac{K_m}{R_a n r_p} \quad ; \quad B_x = \frac{B_{eq}}{n^2 r_p^2} + \frac{K_b K_m}{n^2 r_p^2 R_a}$$

Table 1 describes all variables and parameters of the IP&C plant.. Define as state variables $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = z$ and $x_4 = \dot{z}$ where x_1 and x_3 are measurable

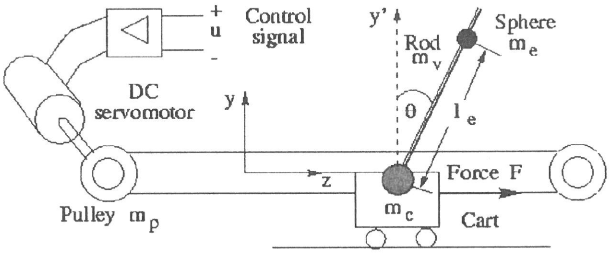

Fig. 1 The PI&C plant (IP view)

variables. Then, the expression of $\dot{x}$ (with $S = \sin x_1$ and $V = \cos x_1$) is found to be

$$\dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ \dfrac{M_2 S x_2^2 - B_x V x_4 - (M_1 + J_2) g S + K_x V K_A u}{(M_2 V^2 - (M_1 + J_2) J_1 / M_2)} \\ x_4 \\ \dfrac{M^2 g S - J_2 M_2 S x_2^2 + J_1 B_x x_4 - J_1 K_x K_A u}{M_2^2 V^2 - (M_1 + J_2) J_1} \end{bmatrix} \tag{4}$$

Application of the Taylor's expansion theorem for the operation point $(\bar{x}, \bar{u})$ into (4) yields

$$\dot{\bar{x}} = f(\bar{x},\bar{u}) + \left[\frac{\partial f(\bar{x},\bar{u})}{\partial x}(x - \bar{x}) + \frac{\partial f(\bar{x},\bar{u})}{\partial u}(u - \bar{u}) \right] + \cdots \tag{5}$$

If we choose $(\bar{x},\bar{u}) = (0,0)$, then equation (5) becomes

$$\dot{\bar{x}} - f(0,0) = \frac{\partial f(0,0)}{\partial x} x + \frac{\partial f(0,0)}{\partial u} u = Ax + Bu \tag{6}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(M_1 + J_2) M_2 g}{(M_1 + J_2) J_1 - M_2^2} & 0 & 0 & \dfrac{B_x M_2}{(M_1 + J_2) J_1 - M_2^2} \\ 0 & 0 & 0 & 1 \\ \dfrac{-M_2^2 g}{(M_1 + J_2) J_1 - M_2^2} & 0 & 0 & \dfrac{-J_1 B_x}{(M_1 + J_2) J_1 - M_2^2} \end{bmatrix} \tag{7}$$

$$B = \begin{bmatrix} 0 \\ \dfrac{-K_x M_2 K_A}{(M_1 + J_2) J_1 - M_2^2} \\ 0 \\ \dfrac{J_1 K_x K_A}{(M_1 + J_2) J_1 - M_2^2} \end{bmatrix} \tag{8}$$

The corresponding discrete-time state-space description of (6) for a sampling time T takes on the form

$$x(k+1) = Gx(k) + Hu(k)\,; \;\; y(k) = Cx(k) \tag{9}$$

where k is the discrete time and y is the output vector. The Crane plant is the Fig. 1 with the rod in resting (stable) position. Following the outlined procedure, then A and B matrices for the Crane plant result

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{-(M_1 + J_2) M_2 g}{(M_1 + J_2) J_1 - M_2^2} & 0 & 0 & \dfrac{-B_x M_2}{(M_1 + J_2) J_1 - M_2^2} \\ 0 & 0 & 0 & 1 \\ \dfrac{-M_2^2 g}{(M_1 + J_2) J_1 - M_2^2} & 0 & 0 & \dfrac{-J_1 B_x}{(M_1 + J_2) J_1 - M_2^2} \end{bmatrix} \tag{10}$$

$$B = \begin{bmatrix} 0 \\ \dfrac{K_x M_2 K_A}{(M_1 + J_2) J_1 - M_2^2} \\ 0 \\ \dfrac{J_1 K_x K_A}{(M_1 + J_2) J_1 - M_2^2} \end{bmatrix} \tag{11}$$

Table 1 *IP&C plant description (m.o.i.r.t. stands for moment of inertia referred to; v.f.c.r.t. stands for viscous friction coefficient referred to).*

Symbol	Description	Value/Relation
l_v	rod length	0.707 m
l_e	distance	Not used
g	Earth gravity	9.8 N.
m_e	sphere mass	Not used
m_v	rod mass	0.063095 kg
m_c	cart mass	0.92 kg
m_p	Pulley mass	0.2 kg
K_A	amplifier gain	14.9
R_a	Armat. resistance	7.38 ohm
L_a	Armat. inductance	Neglected
K_b	back EMF	0.0310352 V/rad
K_m	torque constant	0.031071 N-m/A
J_m	m.o.i.r.t. the motor	0.00000196 kg-m^2
B_m	v.f.c.r.t. the motor	0.001834 N-m/rad/s
J_o	m.o.i.r.t. the load	Neglected
B_o	v.f.c.r.t. the load	Neglected
n	gear ratio	1/19.741
r_p	Radius of the pulley	0.0648 m
J_p	m.o.i. of the pulley	$M_p r_p^2/2$
J_{eq}	equivalent m.o.i.	$J_m + n^2(J_o + J_p)$
B_{eq}	equivalent v.f.c.	$B_m + n^2 B_o$
J_e	m.o.i. of the sphere	$m_e l_e^2$
J_v	m.o.i. of the rod	$m_v l_v^2/3$

THE PI SERVO SYSTEM (SS)

Fig. 2 shows the SS used in this paper, where $r(k)$ is a stepwise reference signal, I is an Identity matrix, z is the shift operator, K is the gain matrix of the state feedback controller, K_I is the gain of the integral controller and $\hat{x}$ is the estimated state vector. For the sequel, let us assume that $\hat{x} = x$. The linear model of the IP&C plant in the Fig. 2 is described by

$$x(k+1) = Gx(k) + Hu(k)\ ;\ y(k) = Cx(k) \quad (12)$$

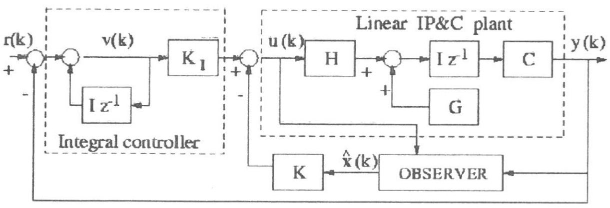

Fig. 2 The PI servo system (SS)

The control law $u(k)$ is found to be

$$u(k) = -K\,x(k) + K_I v(k)\ ;\ K = [K_1, \ldots, K_n] \quad (13)$$

where the variabe $v(k)$ verifies

$$v(k) = v(k-1) + r(k) - y(k) \quad (14)$$

and the output equation takes on the augmented form

$$y(k) = [C \quad 0]\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} = \tilde{C}\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \quad (15)$$

Using the fact that for $k \to \infty$; $r(k) = r(k+1) = \cdots$ then the state-space equation of the SS is found to be

$$\xi(k+1) = \tilde{G}\xi(k) + \tilde{H}w(k)\ ;\ w(k) = -\tilde{K}\xi(k) \quad (16)$$

where

$$x(k) - x(\infty) = x_e(k) \quad (17)$$

$$v(k) - v(\infty) = v_e(k) \quad (18)$$

$$\xi(k) = \begin{bmatrix} x_e(k) \\ v_e(k) \end{bmatrix};\ \tilde{G}(k) = \begin{bmatrix} G & 0 \\ -CG & I \end{bmatrix} \quad (19)$$

$$\tilde{H}(k) = \begin{bmatrix} H \\ -CH \end{bmatrix};\ \tilde{K}(k) = [K \quad -K_I] \quad (20)$$

THE POLE PLACEMENT CONTROLLER (PPC)

The state feedback gain matrix $\tilde{K} = [K \quad -K_I]$ of the PPC can be calculated from [2]

$$\tilde{K} = [0 \quad 0 \quad \ldots \quad 0 \quad 1]M^{-1}\Phi(\tilde{G}) \quad (21)$$

$$\Phi(\tilde{G}) = \tilde{G}^n + \alpha_1\tilde{G}^{n-1} + \cdots + \alpha_{n-1}\tilde{G} + \alpha_n I \quad (22)$$

where n = 5 is the order of the SS (note that the order of the IP&C plant is 4). The controllability matrix M needs to be of full order; that is

$$\text{rank } M = \text{rank}\left[\tilde{H} \quad \tilde{G}\tilde{H} \quad \cdots \quad \tilde{G}^{n-1}\tilde{H}\right] = n \quad (23)$$

The parameters $\alpha_1, \ldots, \alpha_n$ canbe obtained from

$$(z-\mu_1)\cdots(z-\mu_n) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n$$

where $\mu_1,\ldots,\mu_n$ are the desired closed-loop poles of the SS. For both SSs, one with the IP plant and the other with the Crane plant, the desired poles are chosen to be: $\mu_{1,2} = 0.9967 \pm 0.0058i$, $\mu_{3,4,5} = 0.98$

THE STEADY-STATEQUADRATIC OPTIMAL CONTROLLER (SSQOC)

Assuming that: The SS is completely state controllable (i.e., equation (23) is satisfied), that Q is a $n \times n$ positive semi definite real symmetric matrix, and that R is a positive constant, then the minimization of the following quadratic cost function is a positive constant, then the minimization of the following quadratic cost function

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[\xi(k)^T Q \xi(k) + w^2(k)R\right] \tag{24}$$

leads to the steady-state Riccati equation

$$P = Q + \tilde{G}^T P \tilde{G} - \tilde{G}^T P \tilde{H}[R + \tilde{H}^T P \tilde{H}]^{-1} \tilde{H}^T P \tilde{G} \tag{25}$$

and to the steady-state feedback controller $\tilde{K}$

$$\tilde{K} = [P + \tilde{H}^T P \tilde{H}]^{-1} \tilde{H}^T P G \tag{26}$$

Q can be a diagonal matrix. For the IP case R is set to 100 while diagonal elements of Q are set to 200, 0, 100, 0, 0.01. For the Crane case R is set to 1 while diagonal elements of Q are set to 500, 0, 200, 0, 0.1.

THE FULL - ORDER STATE OBSERVER (FOSO)

Fig. 3 depicts the block diagram of the FOSO, where

$$x(k+1) = Gx(k) + H\,u(k); \quad y(k) = Cx(k) \tag{27}$$

$$\tilde{x}(k+1) = (G - K_e C)\tilde{x}(k) + Hu(k) + K_e y(k) \tag{28}$$

Note that K_e is the state observer gain matrix. Define

$$e(k) = x(k) - \tilde{x}(k) \tag{29}$$

Subtracting (28) from (27) and using (29), we can obtain the error equation of the FOSO

$$e(k+1) = (G - K_e C)e(k) \tag{30}$$

The K_e matrix needs to be chosen such that the eigenvalues of the characteristic equation of the FOSO

$$\det[zI - G + K_e C] = 0 \tag{31}$$

make the error $e(k)$ zero with sufficient speed. Such a matrix K_e can be computed from [2]

$$K_e = \Phi(G)N^{-1}[0 \quad 0 \ldots 0 \quad 1]^T \tag{32}$$

$$\Phi(G) = G^m + \alpha_1 G^{m-1} + \ldots + \alpha_{m-1}G + \alpha_m I \tag{33}$$

where $m = 4$ is the order of the linear IP&C plant.The observability matrix N needs to be of full order

$$\text{rank } N = \text{rank } [C^T \; G^T C^T \ldots (G^T)^{m-1} C^T] = m \tag{34}$$

and the parameters $\alpha_1,\ldots,\alpha_n$ can be obtained from

$$(z - \mu_1)\cdots(z - \mu_m) = z^m + \alpha_1 z^{m-1} + \ldots + \alpha_{m-1}z + \alpha_m$$

where $\mu_1,\ldots,\mu_n$ are the desired closed-loop poles of the FOSO. For the IP plant the desired poles are chosen to be; $\mu_{1,2} = 0.7165$, $\mu_3 = 0.9$; $\mu_4 = 0$, while for the Crane plant are: $\mu_{1,2} = 0.1353$, $\mu_3 = 0.9$, $\mu_4 = 0$.

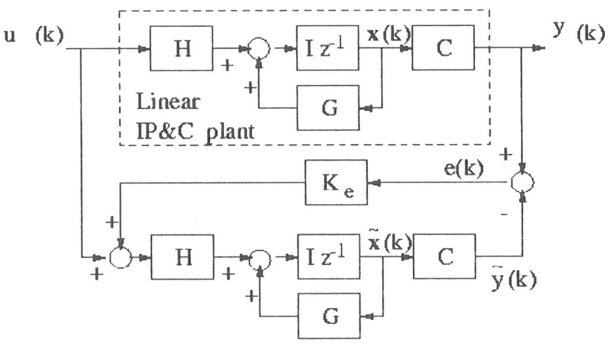

Fig. 3 The full-order state observer (FOSO).

THE MINIMUM-ORDER STATE OBSERVER (MOSO)

Let us consider that the state vector $x(k)$ in (27) can be partitioned into two parts: A p-vector $x_a(k)$ containing the measurable states (in our case,states x_1 and x_3) and a (m-p)-vector $x_b(k)$ holding unmeasurable states. There fore, the partitioned state-space description of the linear IP&C plant becomes

$$\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix} = \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \begin{bmatrix} H_a \\ H_b \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} I & Z \end{bmatrix} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} \qquad (35)$$

where matrices $G_{aa}, G_{ab}, G_{ba}, G_{bb}, H_a, H_b, I$ (the Identity matrix) and Z (the zero matrix) possess proper dimensions. It can be shown [2] that the MOSO feedback gain matrix K_o is given by

$$K_o = \Phi(G_{bb}) \begin{bmatrix} G_{ab} \\ G_{ab}G_{bb} \\ \vdots \\ G_{ab}G_{bb}^{m-p-2} \\ G_{ab}G_{bb}^{m-p-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = O^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (36)$$

$$\Phi(G_{bb}) = G_{bb}^{m-p} + \alpha_1 G_{bb}^{m-p-1} + \cdots + \alpha_{m-p-1} G_{bb} + \alpha_{m-p} I \qquad (37)$$

provided that the rank of matrix O is $m-p$. For our case, $p=2$ Parameters $\alpha_1, \ldots, \alpha_{m-p}$ can be obtained from the characteristic equation of the MOSO

$$|zI - G_{bb} + K_o G_{ab}| = (z-\mu_1)\cdots(z-\mu_{m-p}) =$$

$$z^{m-p} + \alpha_1 z^{m-p-1} + \cdots + \alpha_{m-p-1} z + \alpha_{m-p} \qquad (38)$$

where $\mu_1, \ldots, \mu_{m-p}$ are the desired eigenvalues of the MOSO. For the IP plant, the desired poles are chosen to be: $\mu_{1,2} = 0.7165$, while for the Crane plant are: $\mu_{1,2} = 0.1353$.

THE QUADRATICOPTIMAL STATE OBSERVER (QOSO)

The QOSO employs the configuration of the FOSO depicted in Fig. 3. From equation (31), we can formulate

$$\det[zI - G + K_e C] = \det[zI - G^T + C^T K_e^T] \qquad (39)$$

On using (16), the characteristic equation of the SS is found to be

$$\det[zI - \tilde{G} + \tilde{H}\tilde{K}] \qquad (40)$$

Comparing equations (40) and (39), dual expressions of equations (25) and (26) can be obtained replacing

$\tilde{G}^T$, by G, $\tilde{H}$ by C^T, $\tilde{K}$ by K_e^T, and P by P_e, i.e.,

$$P_e = Q_e + GP_eG^T - GP_eC^T[R_e + CP_eC^T]^{-1}CP_eG^T \qquad (41)$$

$$K_e = [R_e + CP_eC^T]^{-1}GP_eC^T \qquad (42)$$

To determine on-line P_e (or P given by equation (25)), we can use the recursive form of the Riccati equation

$$\begin{aligned} P_e(k+1) = Q_e + GP_e(k)G^T - \\ GP_e(k)C^T[R_e + CP_e(k)C^T]^{-1}CP_e(k)G^T \end{aligned} \qquad (43)$$

R_e and Q_e can be chosen to be diagonal matrices. For the IP case diagonal elements of R_e are set to 1, 10 while for Q_e are set to 1, 0, 1000, 0.9, 1000. For the Crane case diagonal elements of R_e are set to 1, 1 while for Q are set to 1000, 1500, 100, 1500.

HARDWARE AND SOFTWARE OF THE SS

Fig.4 depicts the hardware configuration of the SS, which includes a DC servo motor, a H-type PWM (pulse Width Modulation) amplifier, a sensor block with two optical encoders for sensing signed rod and cart positions, a LabPC+ Input/Output interface card, and a Pentium PC. The information from the sensor block is transmitted in form of two pulse trains, one of them leading the other by $\pi/2$ rad.

The designed SS's were simulated using MATLAB for testing control performance and SS behavior before implementation. All source files for real-time implementation are written in C-code. Unavoidable inherent nonlinearities, like servomotor saturation and coulombic friction, are compensated by software.

EXPERIMENTAL RESULTS

All experimental results were obtained by setting the reference position of the cart to 1.5 m and the sampling frequency to 200 Hz (0.05 ms). Rod up and rod down positions correspond to zero angular positions of the

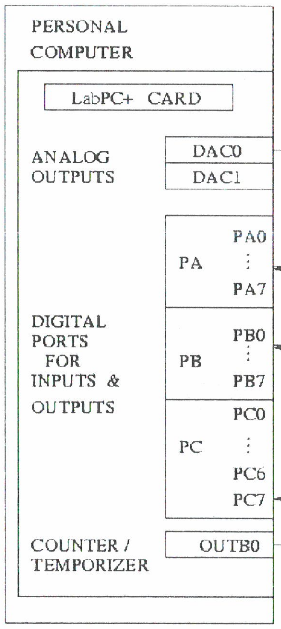

Fig. 4 Hardware implementation of the SS

IP and of the Crane, respectively. The following figures depict cart and rod positions for each designed SS. Recall that the acronyms PPC, SSQOC, FOSO, MOSO, and QOSO stand for pole placement controller, steady-state quadratic optimal controller, full-order state observer, minimum-order state observer, and quadratic optimal state observer, respectively. Observe that all the designed SSs are able to stabilize either the IP or the Crane mounted on a servomotor-driven cart.

CONCLUSIONES

In light of the experimental results, we can assure that it is possible to find a linear process model capable of capturing significant features of the actual nonlinear process. Such a process model permit to employ linear controllers and observers.

Despite the restriction imposed by the linearization procedure, we found that the designed SSs are able to stabilize angular positions of the rod up to 20°.

Experimental results demonstrated that the control performance of the configurations IP+ SSQOC+QOSO (figures 9 and 10) and Crane+SSQOC+QOSO (figures 15 and 16) are superior than the control performance of the other configurations.

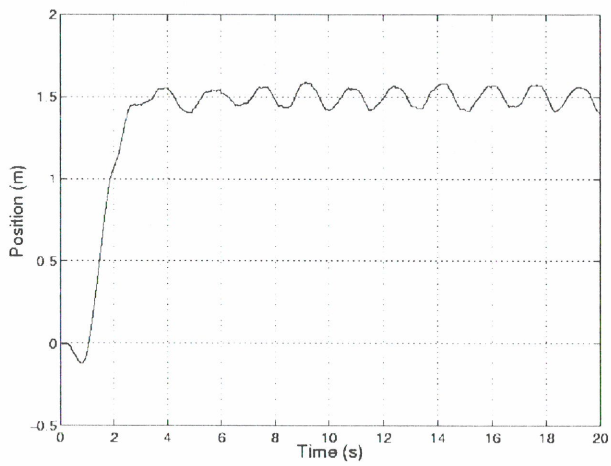

Fig. 5 Cart position (m). SS: IP+PPC+FOSO.

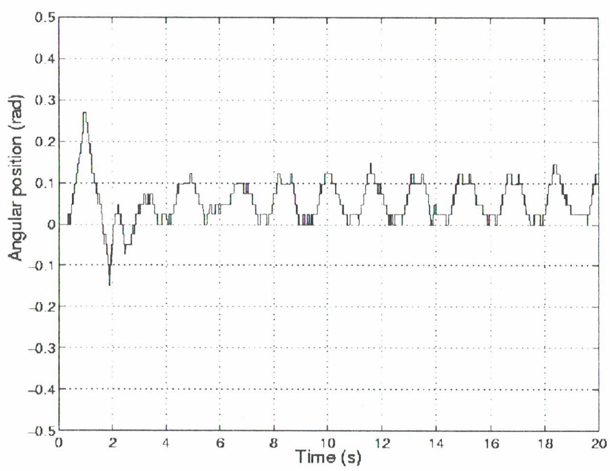

Fig. 6 Rod position (rad). SS: IP+PPC+FOSO.

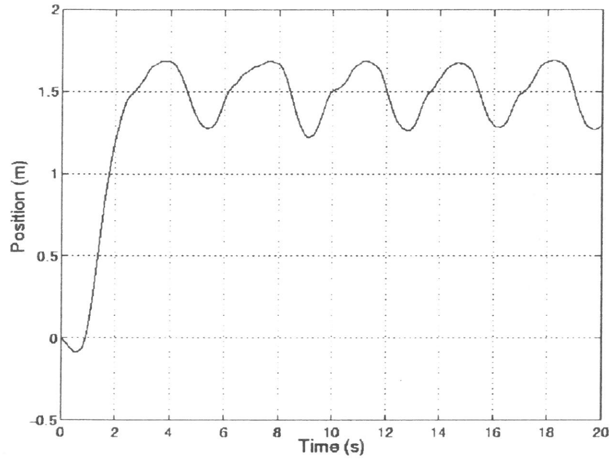

Fig. 7 *Cart position (m). SS: IP+PPC+MOSO.*

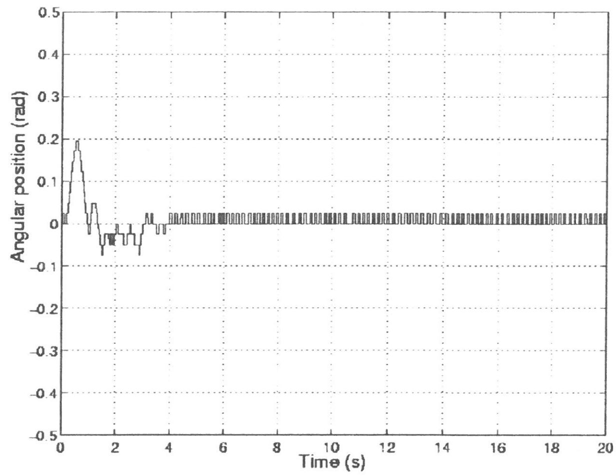

Fig. 10 *Rod position (rad). SS: IP+SSQOC+QOSO.*

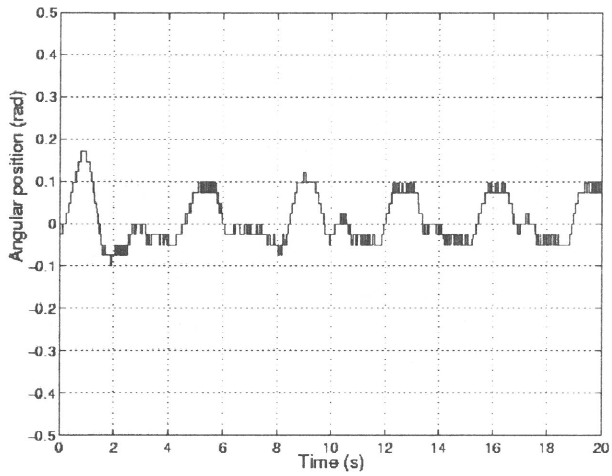

Fig. 8 *Rod position (rad). SS: IP+PPC+MOSO.*

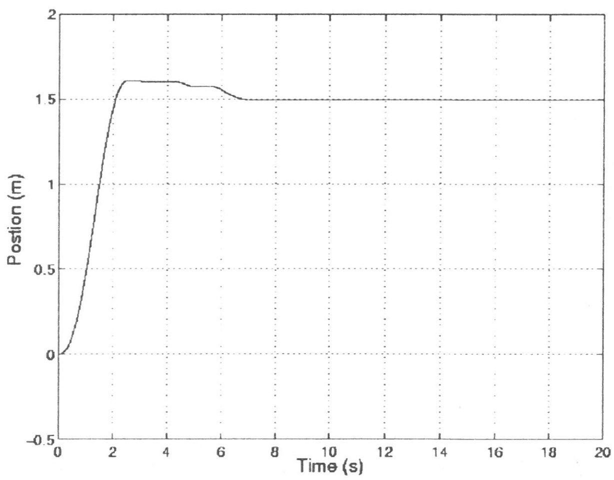

Fig. 11 *Cart position (m). SS: Crane+PPC+FOSO.*

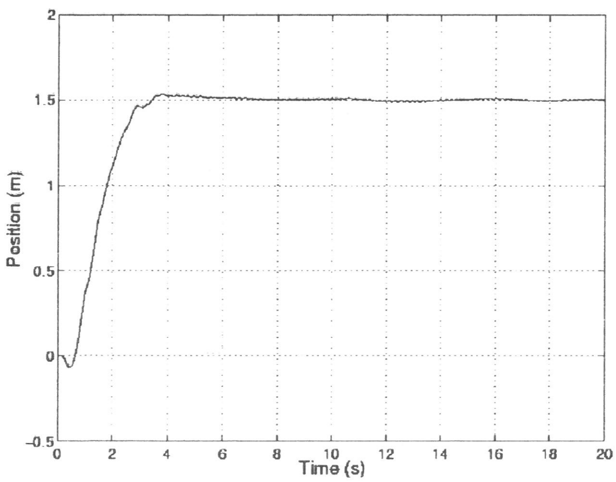

Fig. 9 *Cart position (m). SS: IP+SSQOC+QOSO.*

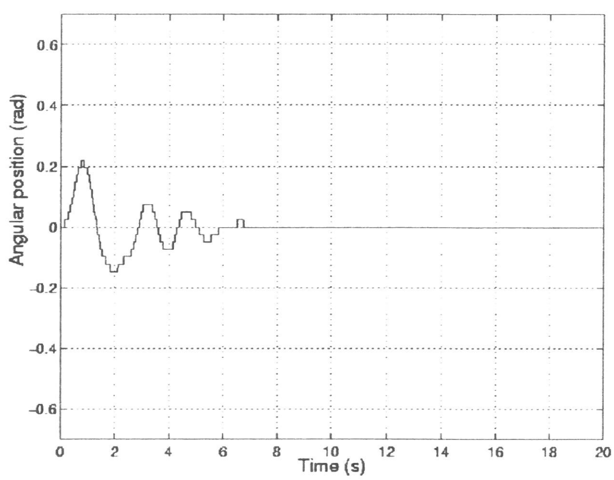

Fig. 12 *Rod position (rad). SS: Crane+PPC+FOSO*

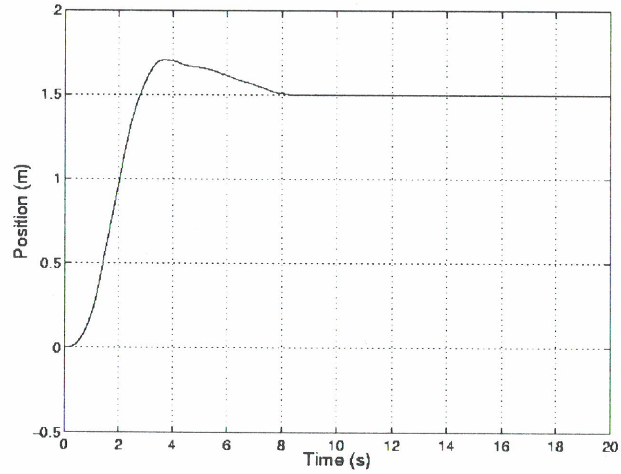

Fig. 13 Cart position (m). SS: Crane+PPC+MOSO.

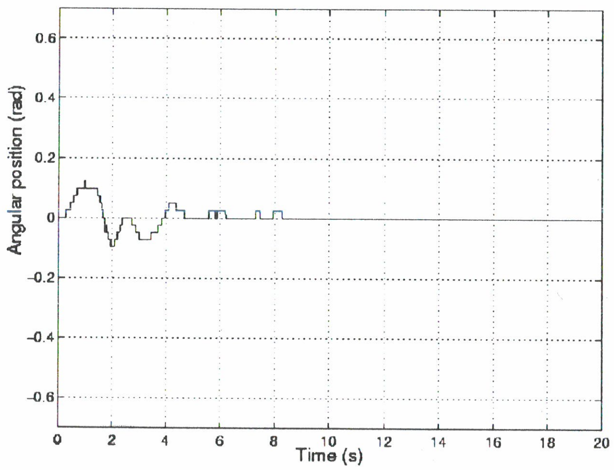

Fig. 14 Rod position (rad). SS: Crane+PPC+MOSO

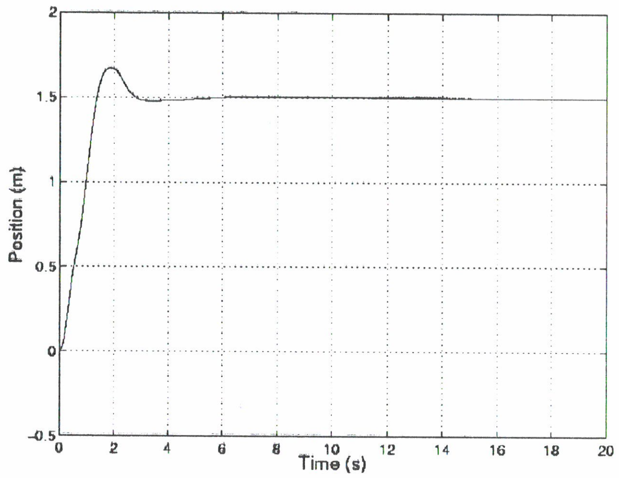

Fig. 15 Cart position (m). SS: Crane+SSQOC+QOSO.

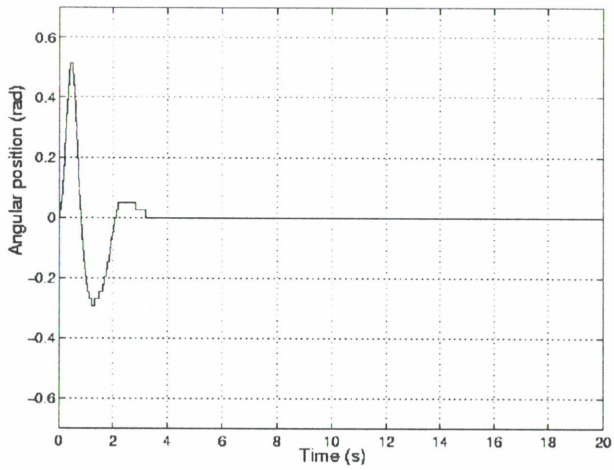

Fig. 16 Rod pos. (rad). SS: Crane+SSQOC+QOSO

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